ME 321: Fluid Mechanics-I

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Lecture - 12 (19/07/2025)
Fluid Dynamics: Flow Measurement
Differential Analysis of Fluid Flow

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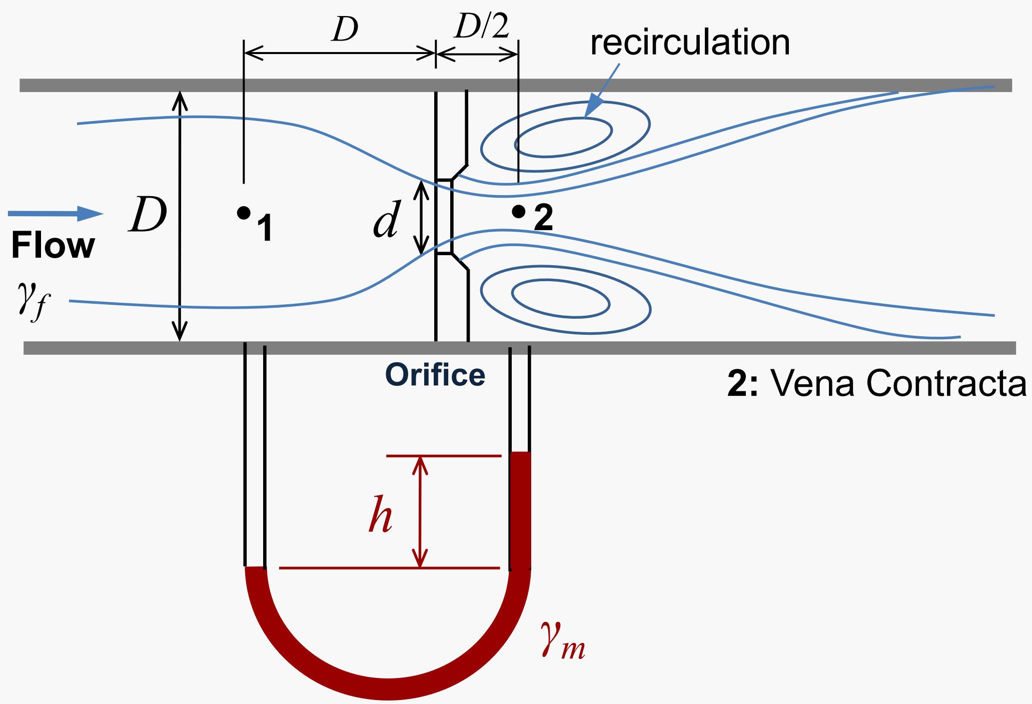
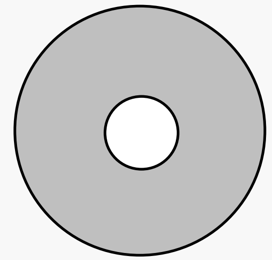
Measurement of

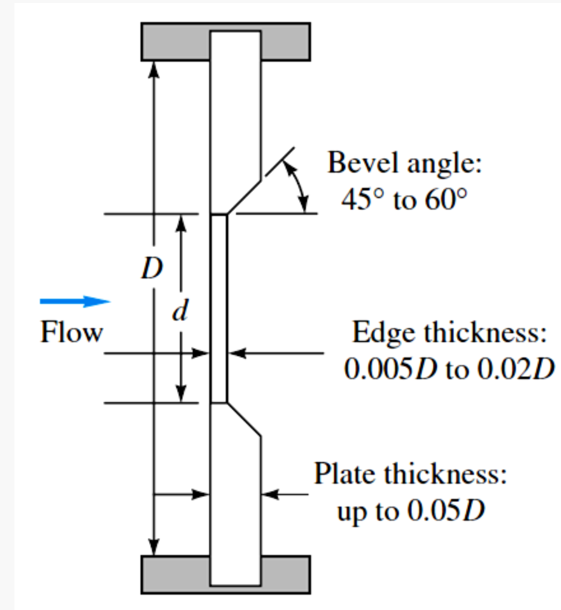
- flow velocity
- flow rate (volume flow rate)

Orifice meter





Orifice plate



ISO standard Orifice plate

Continuity equation: $Q = \frac{\pi}{4} D^2 V_1 = \frac{\pi}{4} {d_2}^2 V_2$

Bernoulli equation between point 1 and 2:

$$\frac{p_1}{\gamma_f} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma_f} + \frac{V_2^2}{2g} + z_2$$

$$\Rightarrow \frac{p_1}{\gamma_f} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma_f} + \frac{V_2^2}{2g}$$

Orifice meter

$$\Rightarrow \frac{V_2^2 - V_1^2}{2g} = \frac{p_1 - p_2}{\gamma_f}$$

$$\Rightarrow V_2^2 - V_1^2 = \frac{2(p_1 - p_2)}{\rho_f}$$

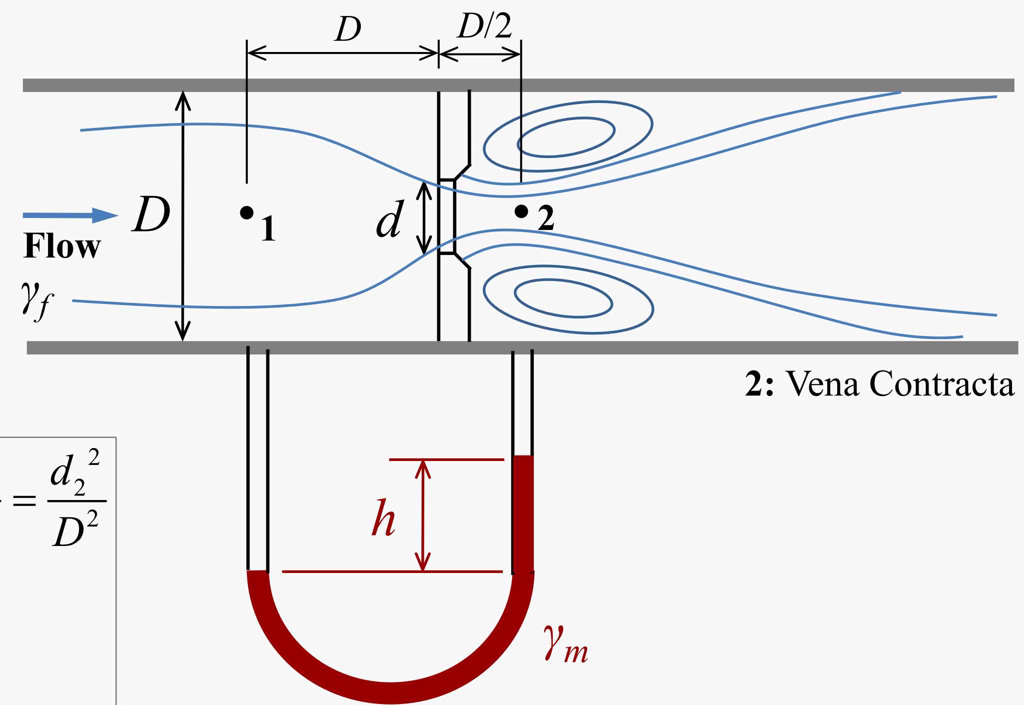
$$\Rightarrow V_2^2\left(1 - \frac{V_1^2}{V_2^2}\right) = \frac{2(p_1 - p_2)}{\rho_f}$$

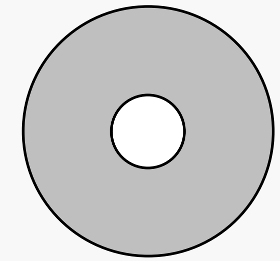
$$\Rightarrow V_2^2\left(1 - \frac{d_2^{\,4}}{D^4}\right) = \frac{2(p_1 - p_2)}{\rho_f}$$

$$\Rightarrow V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho_f\left(1 - \frac{d_2^{\,4}}{D^4}\right)}}$$

$$Q = \frac{\pi}{4} D^2 V_1 = \frac{\pi}{4} d_2^{\,2} V_2 \quad \therefore \frac{V_1}{V_2} = \frac{d_2^{\,2}}{D^2}$$

$$\therefore V_1 = \frac{Q}{\frac{\pi}{4} D^2} \quad \text{and} \quad V_2 = \frac{Q}{\frac{\pi}{4} d_2^{\,2}}$$





Then the theoretical volume flowrate could be expressed as:

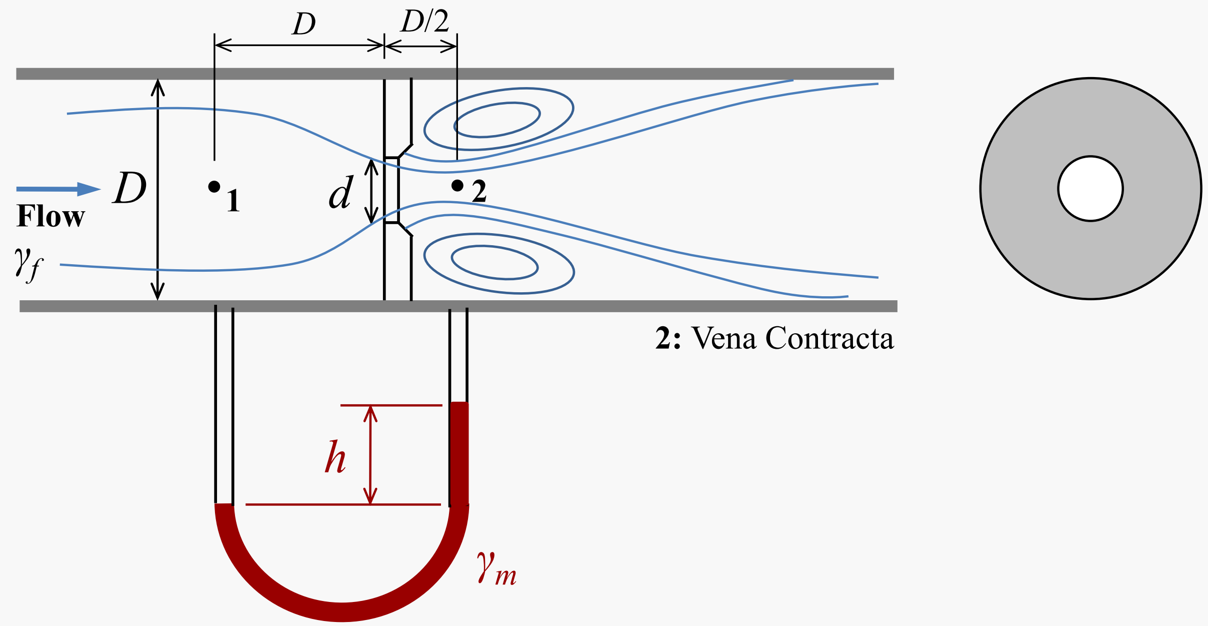
$$Q_{theo.} = A_2 V_2 = \frac{A_2}{\sqrt{\left(1 - \frac{d_2^{\,4}}{D^4}\right)}} \sqrt{\frac{2(p_1 - p_2)}{\rho_f}} \qquad (i)$$

Orifice meter

It is not convenient to use point 2 (vena contracta), instead orifice diameter, *d* is used which could be available from geometric configuration.

Define a coefficient, namely "coefficient of contraction, C_c" as:

$$C_c = \frac{A_2}{A_0} = \frac{d_2^2}{d^2} \left(\equiv \frac{\text{area at vena contracta}}{\text{area at the orifice}} \right)$$



Then Eq. (i) comes as:

$$Q_{theo.} = \frac{C_c A_0}{\sqrt{\left(1 - \frac{C_c{}^2 d^4}{D^4}\right)}} \sqrt{\frac{2(p_1 - p_2)}{\rho_f}} \qquad (ii)$$

However, assuming $C_c \approx 1$, then Eq. (ii) can be written as:

$$Q_{theo.} \approx \frac{A_0}{\sqrt{\left(1 - \frac{d^4}{D^4}\right)}} \sqrt{\frac{2(p_1 - p_2)}{\rho_f}} \qquad (iii)$$

$$Q_{theo.} \approx \frac{A_0}{\sqrt{(1 - \beta^4)}} \sqrt{\frac{2(p_1 - p_2)}{\rho_f}} \qquad (iv)$$

$$\text{where} \quad \beta = \frac{d}{D}$$

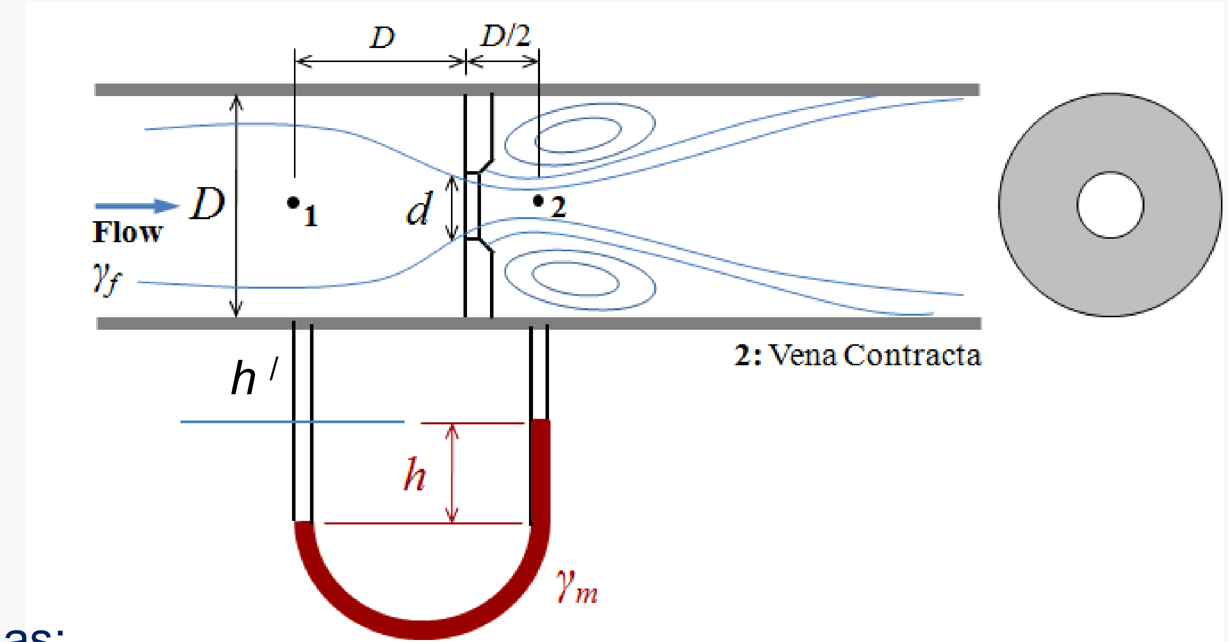
The consideration of vena contracta (point 2) is accommodated by measuring the downstream pressure at point 2 (NOT at the orifice).

Orifice meter

Frictional (viscous) effect becomes very important while such obstruction meter is used in flow systems. To accommodate such effect, empirical **discharge coefficient, C_d** is defined as:

$$C_d = \frac{Q_{actual}}{Q_{theo.}}$$

$$\therefore Q_{actual} = C_d Q_{theo.} = \frac{C_d \; A_0}{\sqrt{\left(1-\beta^4\right)}} \sqrt{\frac{2\left(p_1 - p_2\right)}{\rho_f}} \qquad (v)$$



Pressure differential can be measured using different approaches; for example using U-tube differential manometer as:

$$p_1 + \left(h' + h\right)\gamma_f = p_2 + h'\gamma_f + h\gamma_m$$

$$\Rightarrow p_1 + h\gamma_f = p_2 + h\gamma_m$$

$$\Rightarrow p_1 - p_2 = h\left(\gamma_m - \gamma_f\right)$$

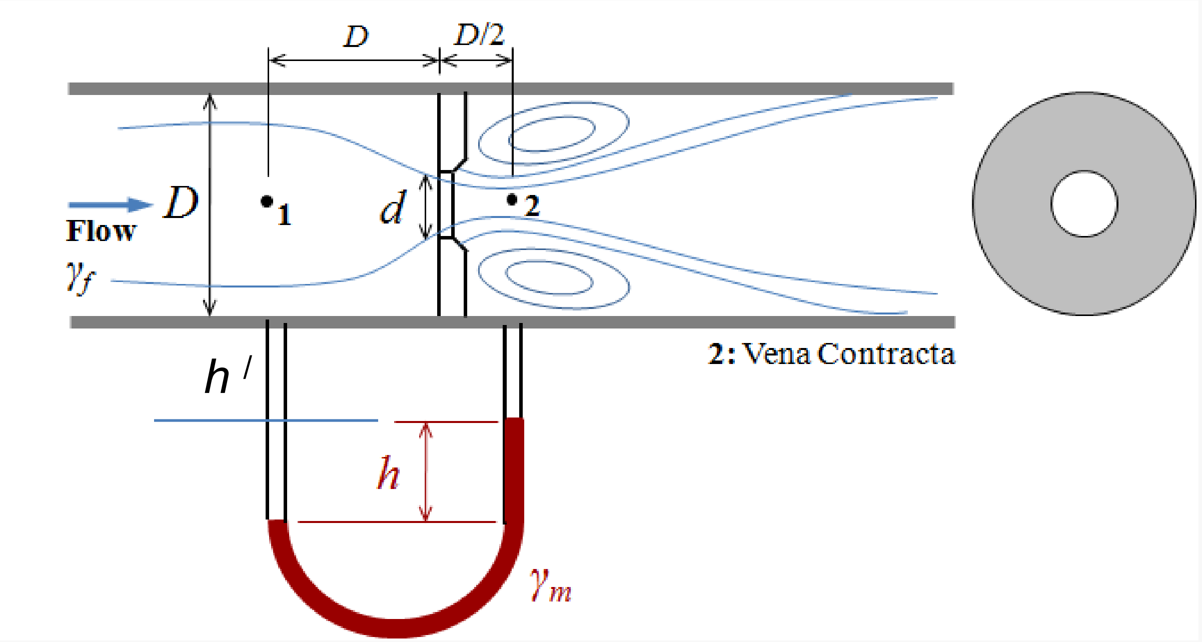
$$\Rightarrow \frac{p_1 - p_2}{\rho_f} = h\left(\frac{\gamma_m - \gamma_f}{\rho_f}\right) \qquad \Rightarrow \frac{p_1 - p_2}{\rho_f} = gh\left(\frac{\rho_m}{\rho_f} - 1\right)$$

specific gravity (SG), S

$$\Rightarrow \frac{p_1 - p_2}{\rho_f} = gh\left(\frac{S_m}{S_f} - 1\right)$$

Orifice meter

$$\therefore Q_{actual} = C_d Q_{theo.} = \frac{C_d\ A_0}{\sqrt{(1-\beta^4)}}\sqrt{\frac{2(p_1-p_2)}{\rho_f}} \qquad (v)$$



$$\therefore Q_{actual} = \frac{C_d\ A_0}{\sqrt{(1-\beta^4)}}\sqrt{2gh\left(\frac{S_m}{S_f}-1\right)}$$

$$\boxed{\therefore Q_{actual} = \frac{C_d\ A_0}{\sqrt{(1-\beta^4)}}\sqrt{2gH}}$$

$$\text{where } H = h\left(\frac{S_m}{S_f}-1\right)$$

S_f is the specific gravity (SG) of the flowing fluid

S_m is the specific gravity (SG) of the manometric fluid

In short form:

$$\boxed{Q_{actual} = K\sqrt{H} \quad ; \quad K = \frac{\sqrt{2g}\ C_d A_0}{\sqrt{(1-\beta^4)}}}$$

Orifice meter

ASME recommends the use of curve-fit formula for C_d developed by ISO according to:

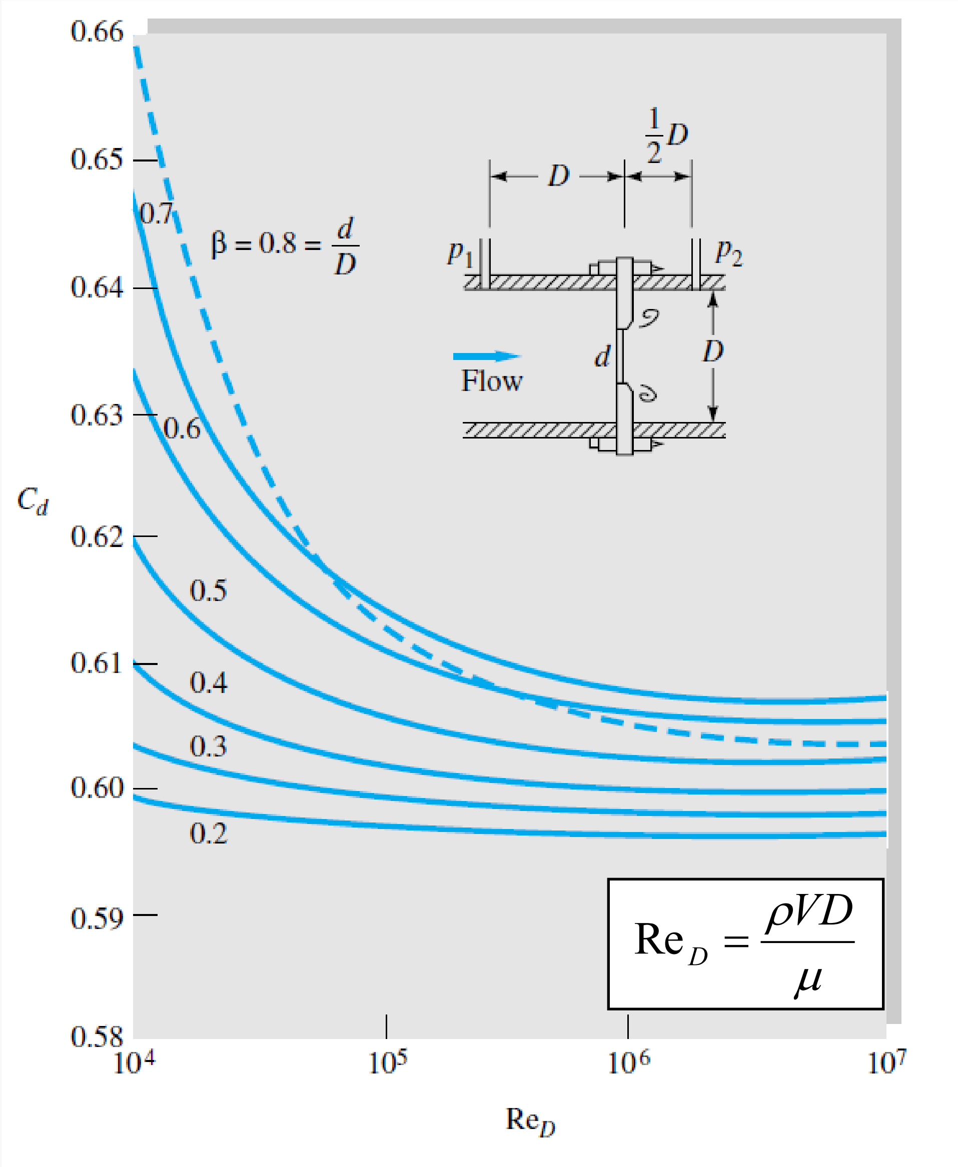
$$C_d = f(\mathrm{Re}_D, \beta)$$

$$C_d = f(\beta) + 91.71\beta^{2.5}\,\mathrm{Re}_D^{-0.75} + \frac{0.09\beta^4}{1-\beta^4}F_1 - 0.0337\beta^3 F_2$$

where:

$$f(\beta) = 0.5959 + 0.0312\beta^{2.1} - 0.184\beta^8$$

$$F_1 = 0.4333 \qquad F_2 = 0.47 \qquad \left(D : \tfrac{1}{2}D \text{ taps}\right)$$



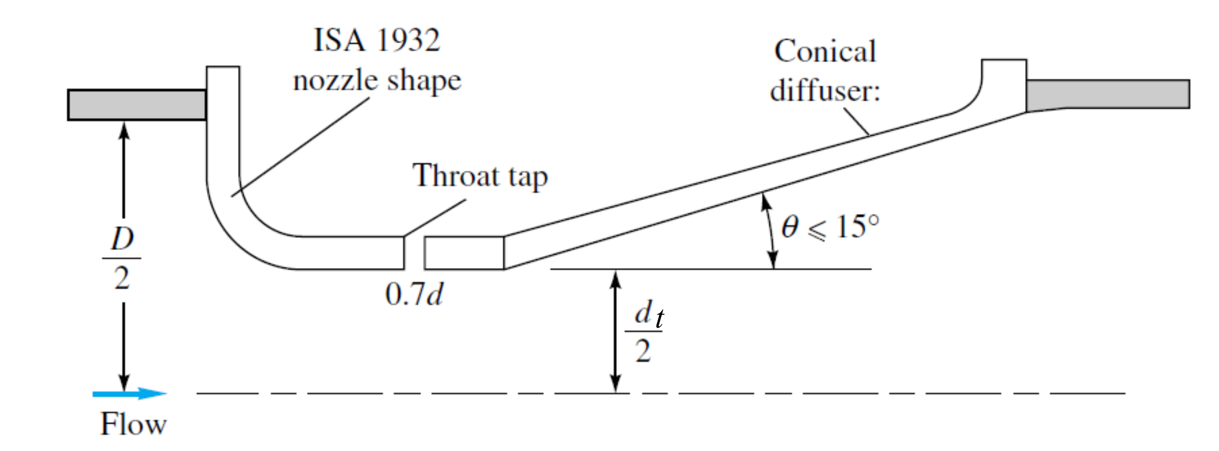
Discharge coefficient of an orifice in the range of Reynolds number 10^4 to 10^7

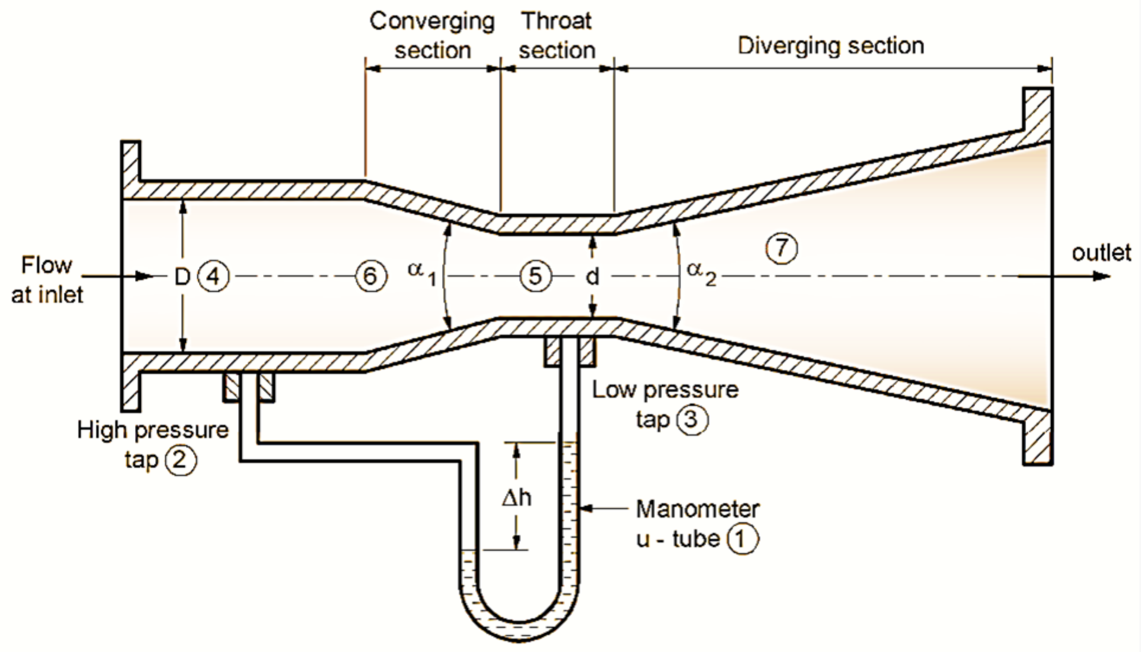
Venturi meter

Modern venuri meter consists of an ISA 1932 nozzle entrance and a conical expansion of half angle no greater than 15 deg. Its discharge coefficient is given by the ISO correlation formula:

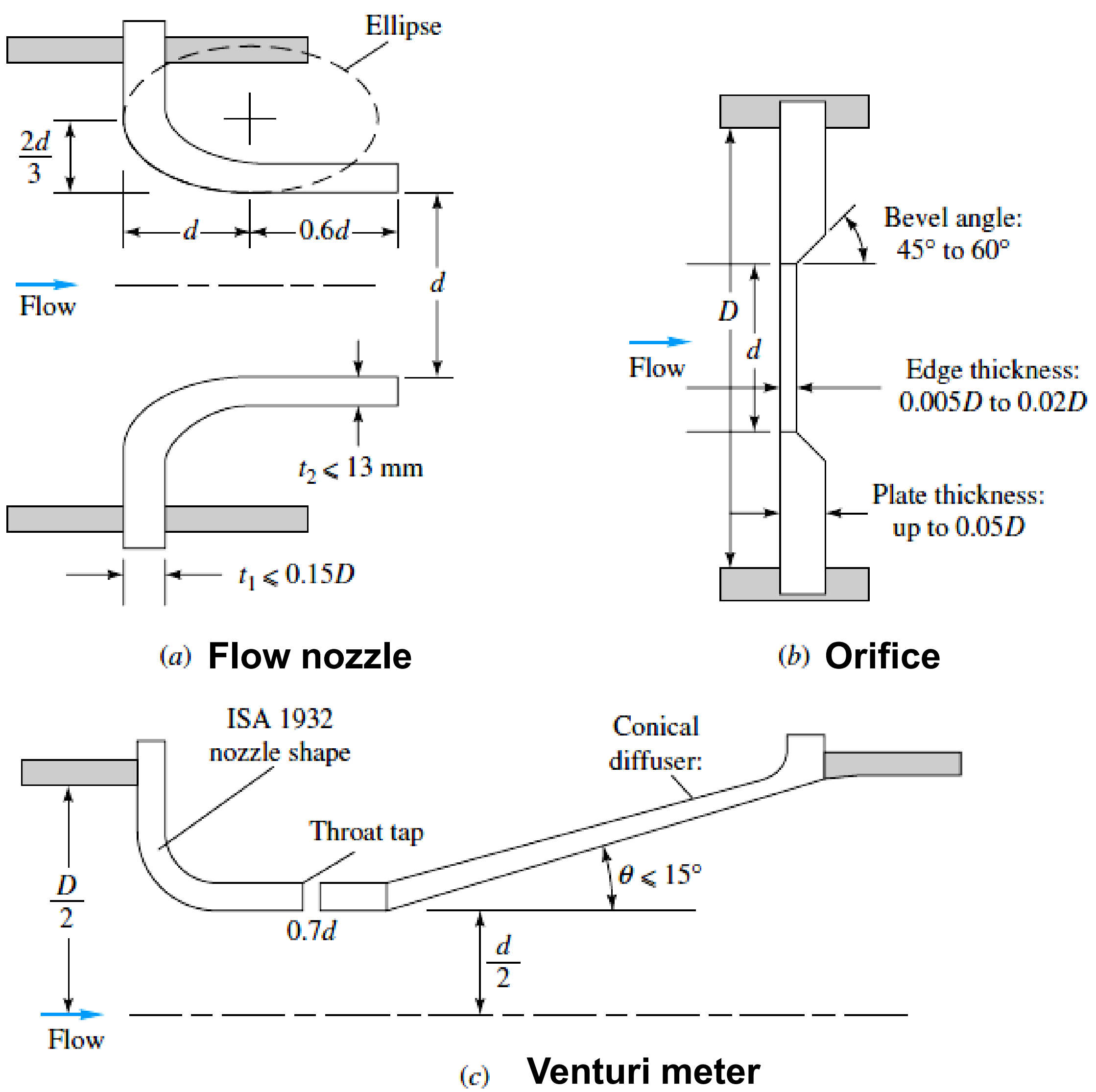
$$C_d \approx 0.9858 - 0.196\beta^{4.5}$$

$$\beta = \frac{d_t}{D}$$





Flow meters



Type of meter	Net head loss	Cost
Orifice	Large	Small
Nozzle	Medium	Medium
Venturi	Small	Large

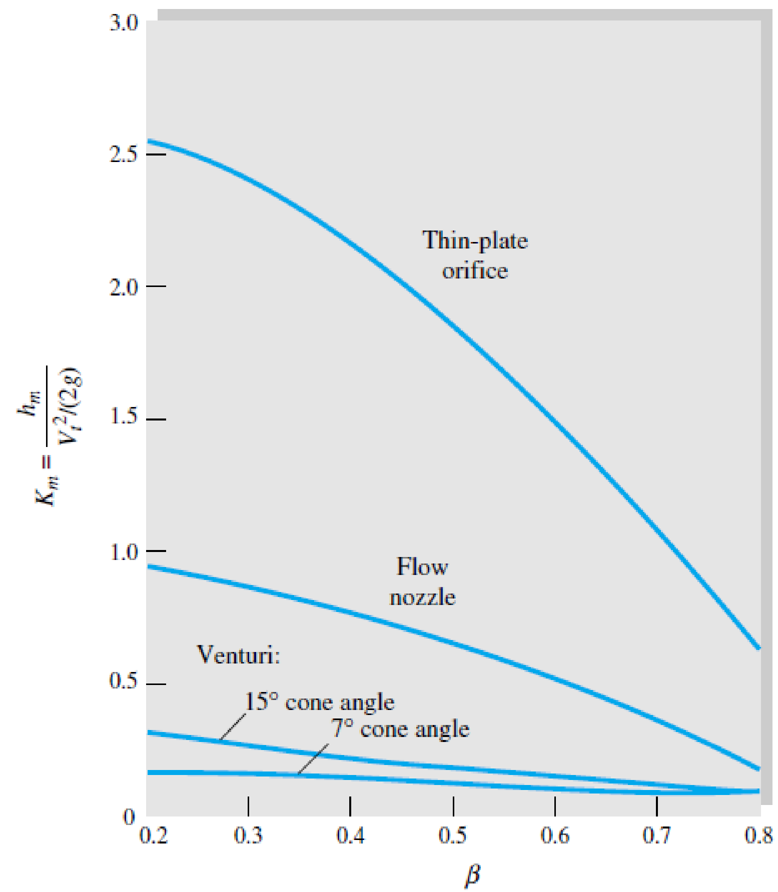


Fig. 6.43 Nonrecoverable head loss in Bernoulli obstruction meters. *(Adapted from Ref. 30.)*

Open Channel Flow measurement

Rectangular notch of size L x H:

Velocity at the infinitesimal element is: $v = \sqrt{2gh}$

Flowrate through the infinitesimal element is:

$$dQ_{theo.} = (Ldh)\sqrt{2gh}$$

Then the actual discharge through the element becomes:

$$dQ_{actual} = C_d dQ_{theo.} = C_d (Ldh)\sqrt{2gh}$$

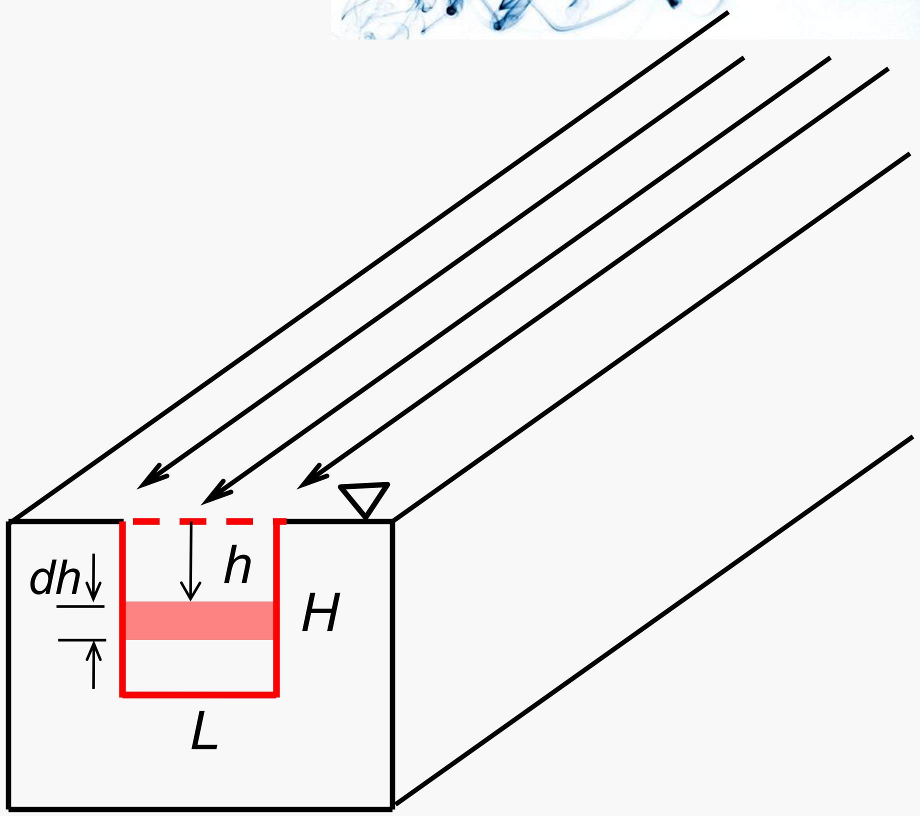
Then the total actual discharge :

$$Q_{actual} = \int_0^H dQ_{actual}$$

$$\Rightarrow Q_{actual} = \int_0^H C_d (Ldh)\sqrt{2gh}$$

$$\Rightarrow Q_{actual} = C_d L \int_0^H \sqrt{2gh}\, dh$$

$$\therefore Q_{actual} = \frac{2}{3}\sqrt{2g}\, C_d L H^{\frac{3}{2}}$$



Open Channel Flow measurement

Triangular notch :

Width of the strip: $2(H-h)\tan\left(\frac{\theta}{2}\right)$

Velocity at the infinitesimal element is: $v=\sqrt{2gh}$

Flowrate through the infinitesimal element is:

$$dQ_{theo.}=\left(2(H-h)\tan\left(\frac{\theta}{2}\right)dh\right)\sqrt{2gh}$$

Then the actual discharge through the element becomes:

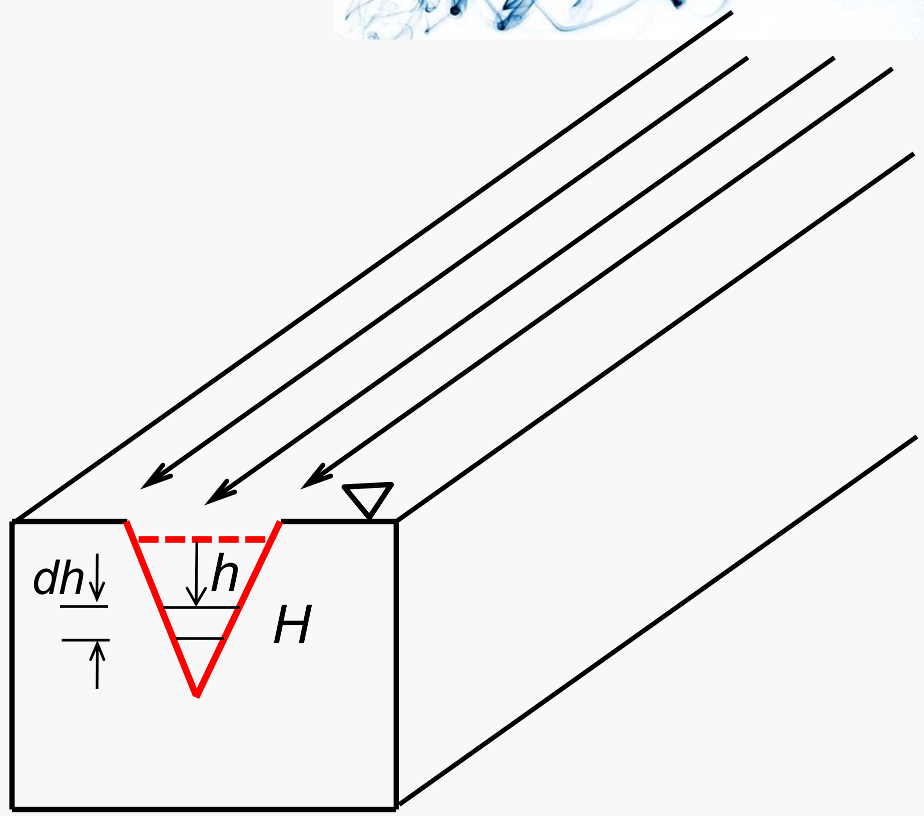
$$dQ_{actual}=C_d\,dQ_{theo.}=C_d\,2(H-h)\tan\left(\frac{\theta}{2}\right)dh\sqrt{2gh}$$

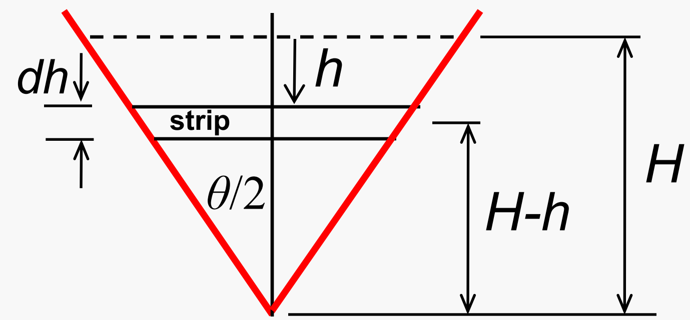
Then the total actual discharge :

$$Q_{actual}=\int_0^H dQ_{actual}$$

$$\Rightarrow Q_{actual}=C_d\int_0^H 2(H-h)\tan\left(\frac{\theta}{2}\right)dh\sqrt{2gh}$$

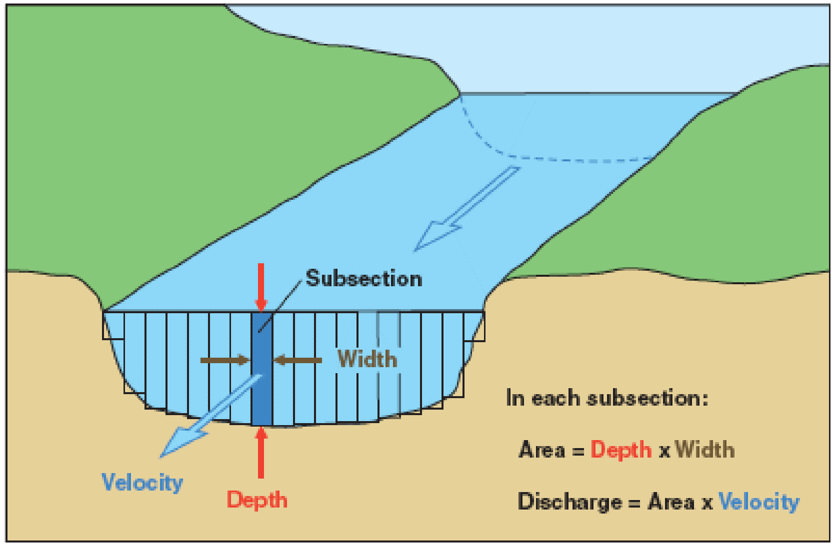
$$\therefore Q_{actual}=\frac{8}{15}\sqrt{2g}\,C_d\,\tan\left(\frac{\theta}{2}\right)H^{\frac{5}{2}}$$





Symmetric triangular notch

Current Flow meter



Current-meter discharge measurements are made by determining the discharge in each subsection of a channel cross section and summing the subsection discharges to obtain a total discharge.

Integral Analysis

While an estimation of **gross effects** (mass flow, induced force, energy change) over a **finite region** or **control volume (CV)** are required, the **integral approach** is adopted. The basic equations of fluid dynamics in **integral form** for a **finite control volume (CV)** are:

(1) Continuity equation:

$$\text{volume flow rate, } Q = AV = \text{constant (for incompressible flow)}$$

$$\text{mass flow rate, } \dot{m} = \rho AV = \text{constant (for any flow: compressible/incompressible)}$$

(2) Bernoulli Equation: (liner momentum equation)

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant (Bernoulli constant) (for incompressible flow)}$$

However, the integral approach does not enable us to obtain the detailed **point-by-point information** of the flow field. For example, the integral approach could provide information on the lift generated by a wing; it could not be used to determine the **pressure and shear stress distributions** that produce the lift on the wing.

Differential Analysis

Once the **point-by-point details** of a flow pattern are required, we need to analyze an ***infinitesimal region*** (small scale/elemental) of the flow. In this case, an ***infinitesimally small control volume*** (in contrast to finite control volume) is taken to apply the basic conservation laws. This approach is known as **differential analysis** of fluid motion.

The analysis yields the basic ***differential equations*** of fluid motion. Appropriate boundary conditions are required to solve these equations. In their most basic form, these differential equations of motion are quite difficult to solve, and very little is known about their general mathematical properties.

However, an approximation technique known as **computation fluid dynamics (CFD)** has been developed whereby the derivatives are simulated by algebraic relations between a finite number of grid points in the flow field, which are then solved on computer.

Conservation of Mass

formulation in differential approach

Conservation of Mass

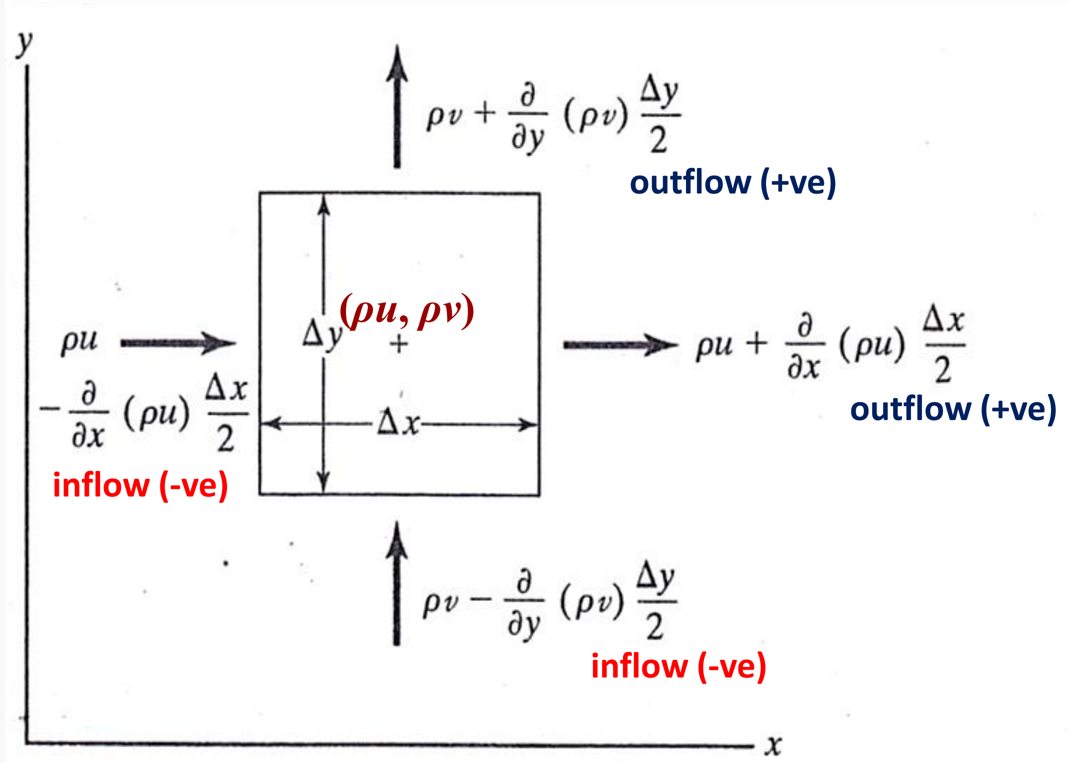
Mass can neither be created nor destroyed.

Consider a very small volume of space (**infinitesimal control volume**) through which a fluid is flowing. For simplicity, a 2D flow is considered and the control volume is bounded by the surfaces Δx and Δy as shown in figure. According to the law,

> **the net outflow of mass through the surfaces surrounding the volume must be equal to the decrease of mass within the volume.**

The mass flow rate is equal to the product of density, velocity component normal to surface and the area of that surface. In vector form;

$$\dot{m} = \int_s \rho(\vec{\mathrm{V}} \cdot \hat{\mathrm{n}})\, dA$$



e **2.3** Velocities and densities for the mass-flow balance gh a fixed volume element in two dimensions.

Conservation of Mass

A first-order Taylor series is used to evaluate the flow properties at the faces of the element, since the properties are a function of position (***continuum approach*** **(Lecture-1, 2)**).

Taylor series

$$f(x+h) = f(x) + h\frac{\partial f(x)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x)}{\partial x^2} + \dots\dots\dots$$

The net outflow of mass per unit of time **per unit depth** is

outflow (+ve) — area — outflow (+ve) — area

$$\left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right](\Delta y \times 1) + \left[\rho v + \frac{\partial(\rho v)}{\partial y}\frac{\Delta y}{2}\right](\Delta x \times 1)$$

$$-\left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right](\Delta y \times 1) - \left[\rho v - \frac{\partial(\rho v)}{\partial y}\frac{\Delta y}{2}\right](\Delta x \times 1)$$

inflow (-ve) — inflow (-ve)

$$\Rightarrow \frac{\partial(\rho u)}{\partial x}\Delta x \Delta y + \frac{\partial(\rho v)}{\partial y}\Delta x \Delta y \qquad \text{(A)}$$

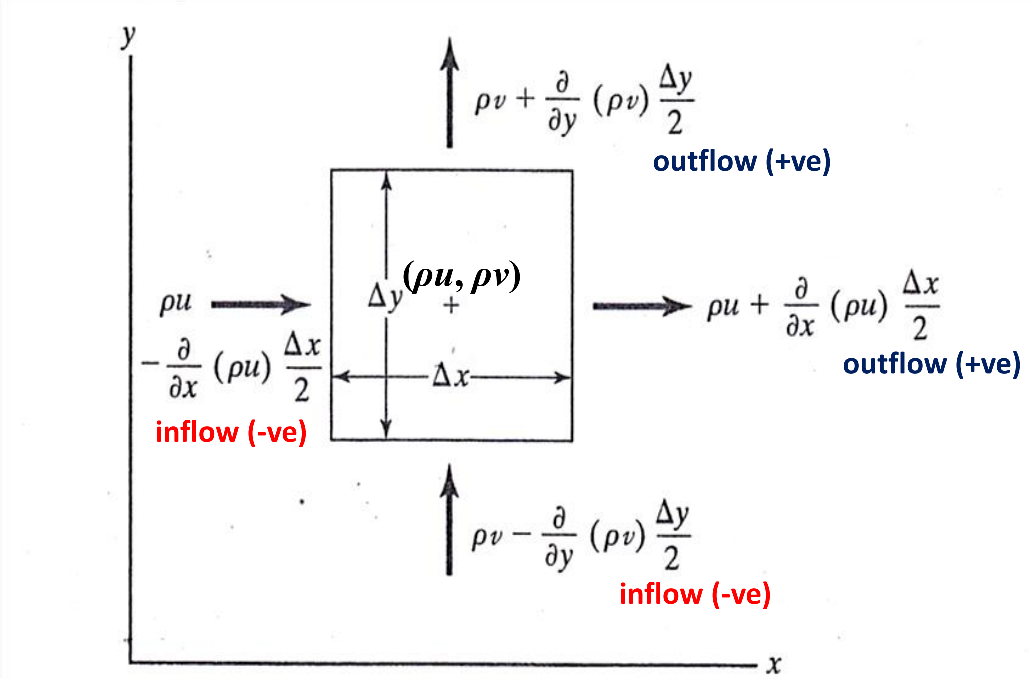


Figure 2.3 Velocities and densities for the mass-flow balance through a fixed volume element in two dimensions.

Conservation of Mass

which must be equal **to the rate at which the mass contained within the element decreases**:

$$-\frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\times 1)=-\frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y) \quad ;\ (-\text{ve due to decrease in mass}) \qquad \text{(B)}$$

mass= density × volume

Equating the above two expressions (**A** & **B**) and dividing by $\Delta x\Delta y$-

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

If z-dimension is considered, the differential form of the above expression comes as

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

$$\Rightarrow \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0 \quad ;\quad \text{where}\ \ \vec{V}=(u,v,w)\ \ \text{and del operator},\ \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

which is known as **differential continuity equation**.

Conservation of Mass

In case of steady flows,

$$\frac{\partial}{\partial t}(\,) = 0$$

Then the **steady flow continuity equation** in differential form becomes as-

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\Rightarrow \quad \nabla \cdot (\rho \vec{V}) = 0$$

$$\Rightarrow \quad \text{div}\,(\rho \vec{V}) = 0$$

Compressible flows ($\rho \neq$ constant)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \quad \nabla \cdot \vec{V} = 0$$

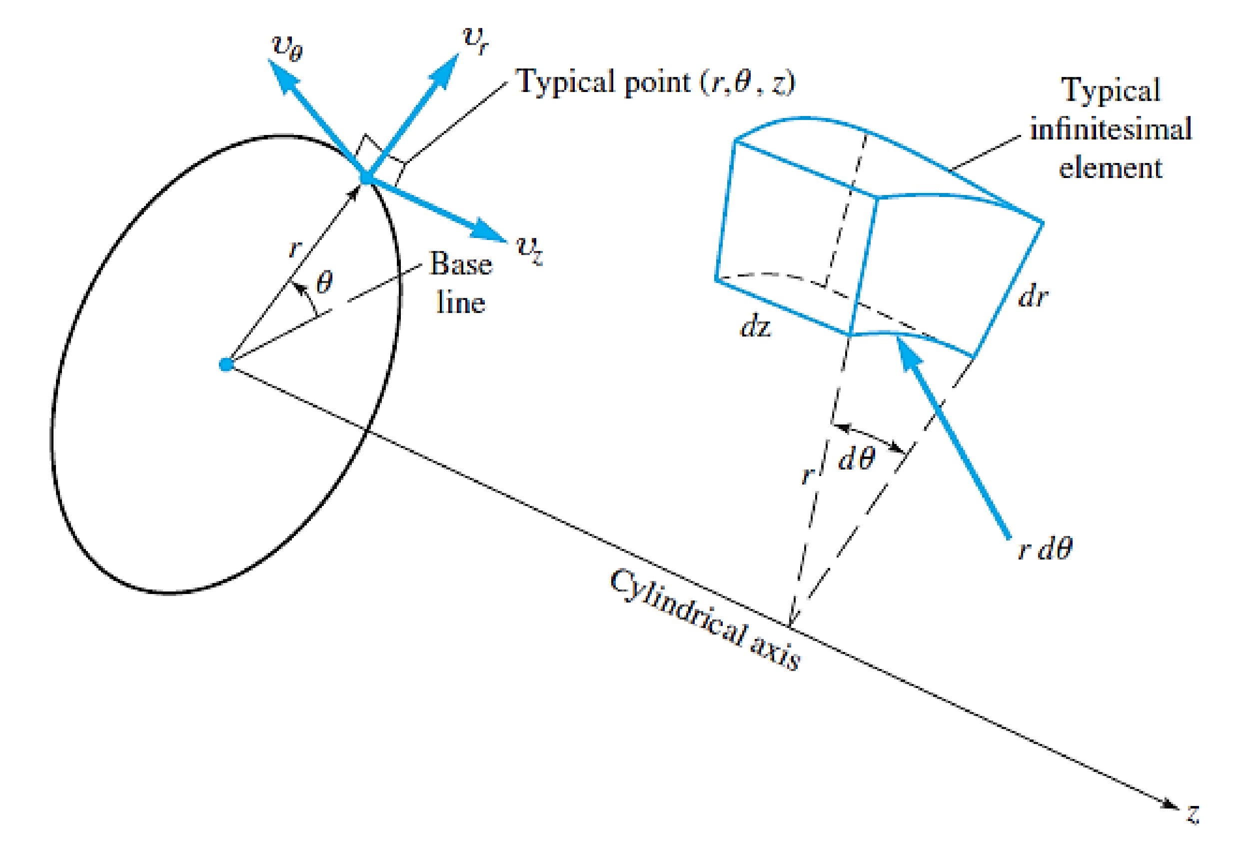
$$\Rightarrow \quad \text{div}\,\vec{V} = 0$$

Incompressible flows ($\rho =$ constant)

Cylindrical Polar Coordinates (r , θ , z)

The divergence of any vector function $\mathbf{A}\,(\,r\,,\theta\,,z\,,t\,)$ is found by making the transformation of coordinates

$$r = (x^2 + y^2)^{1/2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad z = z$$



$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(A_z)$$

Then, the steady flow continuity equation **in differential form in cylindrical coordinate system:**

$$\nabla \cdot (\rho \vec{V}) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$