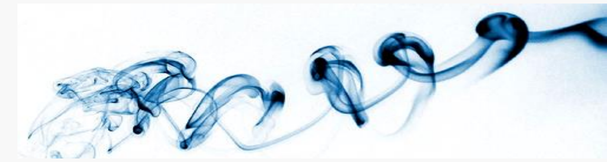

ME 321: Fluid Mechanics-I

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Lecture - 13 (26/07/2025)
Differential Analysis of Fluid Flow

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Conservation of Linear Momentum

formulation in differential approach

Conservation of linear momentum

Linear Momentum Equation:

The net force acting on a fluid particle is equal to the time rate of change of the linear momentum of the fluid particle.

As fluid element moves in space, its velocity, density, shape and volume may change, but its mass is conserved. Conservation of momentum can be written as-

$$\vec{F} = m\frac{D\vec{V}}{Dt} \quad ; \quad \vec{V} = (u, v, w) \quad \text{and} \quad \vec{F} = (F_x, F_y, F_z)$$

$$\left.\begin{aligned} x-\text{direction}: \quad & F_x = m\frac{Du}{Dt} \\ y-\text{direction}: \quad & F_y = m\frac{Dv}{Dt} \\ z-\text{direction}: \quad & F_z = m\frac{Dw}{Dt} \end{aligned}\right\} \qquad (1)$$

The velocity of a fluid particle is, in general, an explicit function of time *t* as well as of its position (*x*, *y*, *z*). Furthermore, the position coordinates *x*, *y*, *z* of the fluid particle are themselves a function of time. The derivative in the above expression is frequently termed as ***particle, total*** *or* ***substantial derivative (D/Dt)*** of velocity **(Lecture - 3: particle acceleration)**.

Conservation of linear momentum

Recap (L-3)

Since

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$\Rightarrow \underbrace{\frac{Du}{Dt}}_{\text{total}} = \underbrace{\frac{\partial u}{\partial t}}_{\text{local}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective}} \; ; \qquad u = \frac{\partial x}{\partial t}, \; v = \frac{\partial y}{\partial t}, \; w = \frac{\partial z}{\partial t}$$

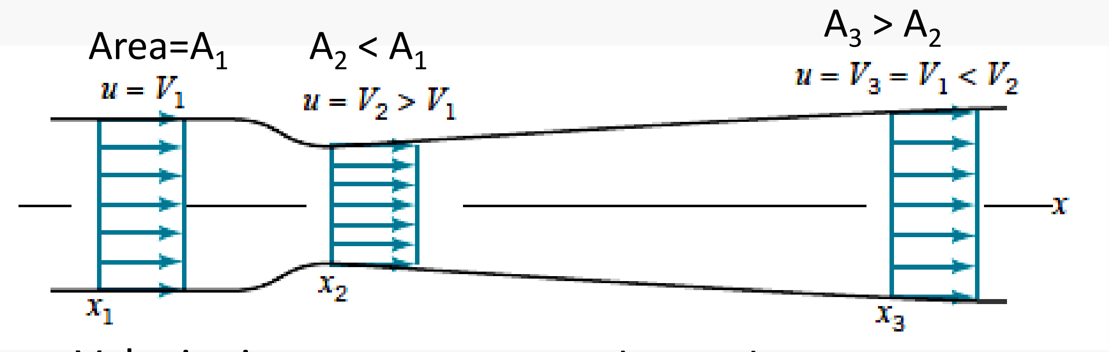
Similarly

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Steady flow

$$\frac{\partial ()}{\partial t} = 0$$



Velocity increases (1) to (2)

Velocity decreases (2) to (3)

$$\boxed{a_x = u\frac{\partial u}{\partial x} \neq 0}$$

Convective acceleration

Conservation of linear momentum

The principal forces with which we are concerned are those which act directly on the mass of the fluid element, the ***body force***, and those which act on its **surface**, the ***pressure forces*** and ***shear forces***. The stress system acting on an element of the surface is shown in figure:

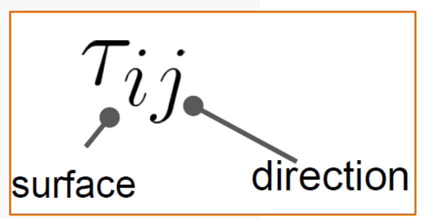
There is a total of 6 shear stresses and 3 normal stresses acting on a fluid element.

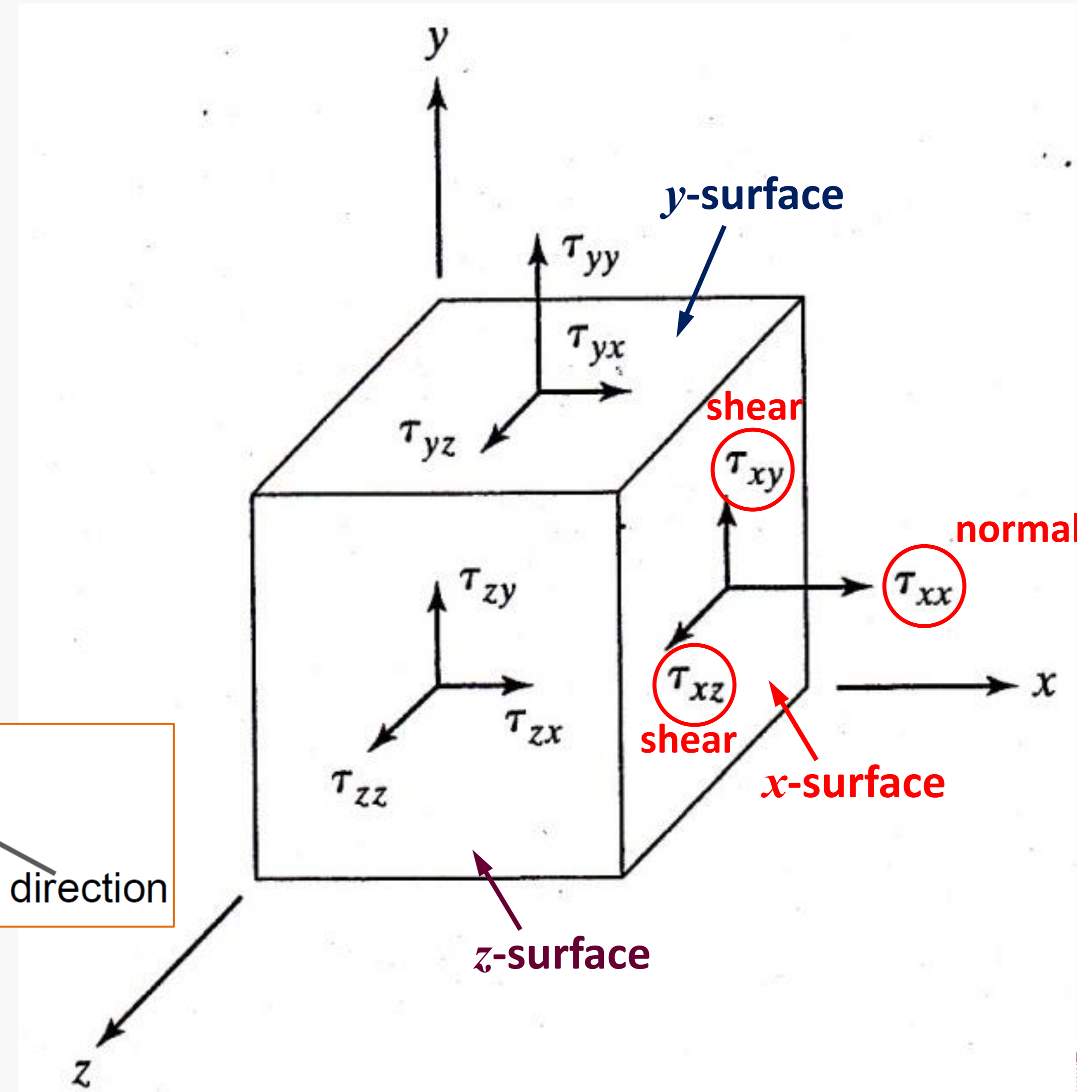
The properties of most fluids have no preferred direction in space; that is, **fluids are isotropic**. As a result-

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{zx} = \tau_{xz}$$





Conservation of linear momentum

In general, the various stresses change from point to point (**continuum approach**). Thus, they produce net forces on the fluid particle, which cause it to accelerate. To simplify the illustration of the force balance on the fluid particle, consider a 2D flow, as indicated in figure.

The resultant force in x-direction (for a unit depth in the z-direction) is

$$\rho f_x \Delta x \Delta y + \frac{\partial}{\partial x}(\tau_{xx})\Delta x \Delta y + \frac{\partial}{\partial y}(\tau_{yx})\Delta x \Delta y$$

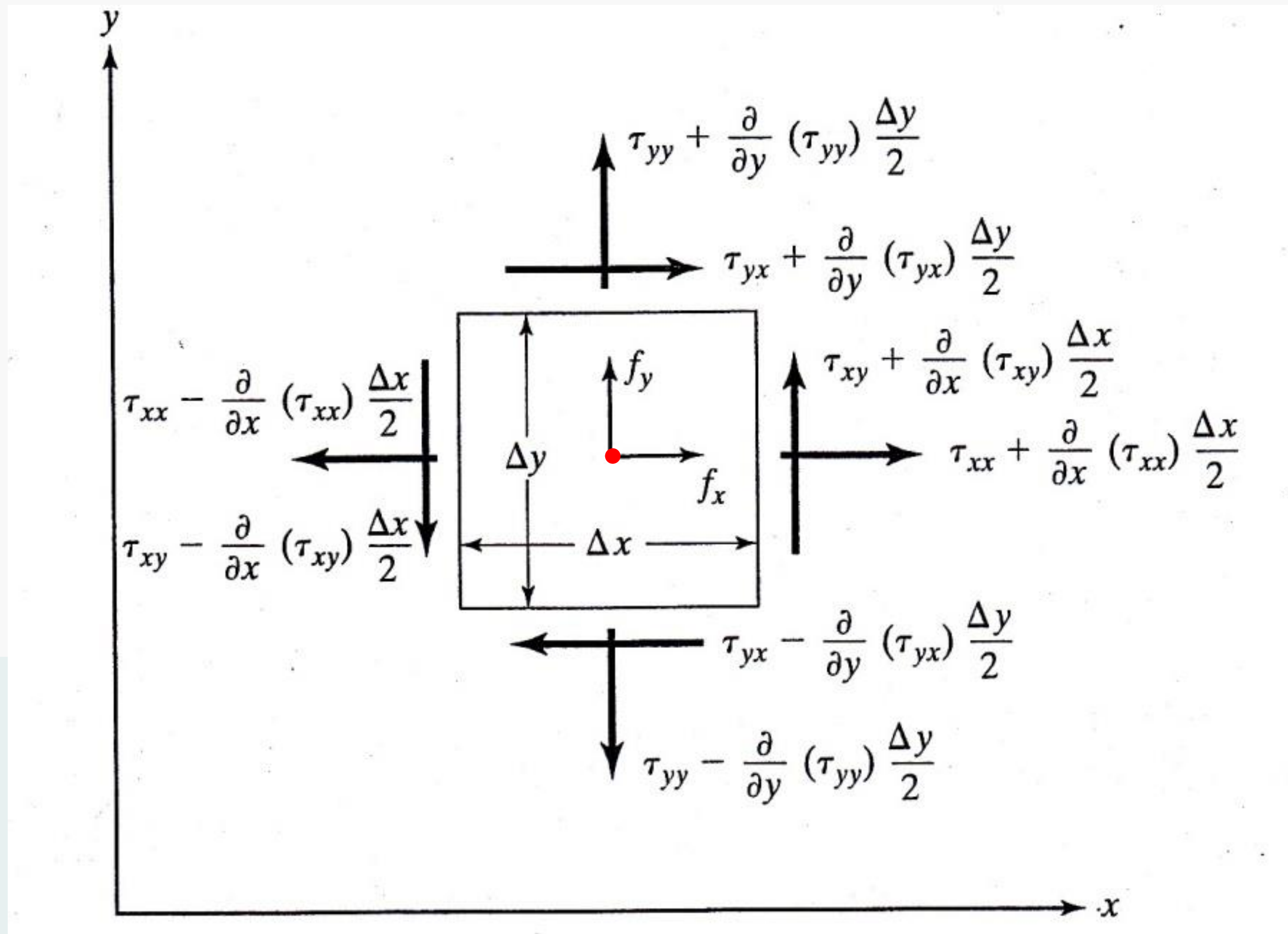


Figure 2.7 Stresses acting on a two-dimensional element of fluid.

where f_x is the body force per unit mass in x-direction.

Including flow in the z-direction, the resultant force in the x-direction-

$$F_x = \rho f_x \Delta x \Delta y \Delta z + \frac{\partial}{\partial x}(\tau_{xx})\Delta x \Delta y \Delta z + \frac{\partial}{\partial y}(\tau_{yx})\Delta x \Delta y \Delta z + \frac{\partial}{\partial z}(\tau_{zx})\Delta x \Delta y \Delta z$$

Conservation of linear momentum

Use this expression in eqn. (1) for x-direction:

$$F_x = \rho f_x \Delta x \Delta y \Delta z + \frac{\partial}{\partial x}(\tau_{xx})\Delta x \Delta y \Delta z + \frac{\partial}{\partial y}(\tau_{yx})\Delta x \Delta y \Delta z + \frac{\partial}{\partial z}(\tau_{zx})\Delta x \Delta y \Delta z = \rho \Delta x \Delta y \Delta z \times \frac{Du}{Dt}$$

$$\Rightarrow \rho \frac{Du}{Dt} = \rho f_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\boxed{\Rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}$$

$$\left.\begin{aligned} x-\text{direction}: \quad F_x &= m\frac{Du}{Dt} \\ y-\text{direction}: \quad F_y &= m\frac{Dv}{Dt} \\ z-\text{direction}: \quad F_z &= m\frac{Dw}{Dt} \end{aligned}\right\} \quad (1)$$

Similarly, for y- and z-directions

$$\boxed{\Rightarrow \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}}$$

$$\boxed{\Rightarrow \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho f_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}}$$

These are the basic forms of **Navier-Stokes equations (NS equations).** Other fluid mechanical relations are obvious to solve such equations.

NS equations are the most famous equations for advanced analysis in fluid dynamics.

Navier-Stokes Equations

Stress-deformation(strain rate) relation:

For incompressible Newtonian fluids it is known that the viscous stresses are related to the rates of deformation (strain rate/**velocity gradient**) and coefficient of viscosity. These can be expressed in Cartesian coordinates (x, y, z) as:

(for 3D, expressions of strain rate is a bit more complicated and details can be found in advanced course of fluid mechanics)

For normal stresses (3D):

$$\tau_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

normal shear

where p is the hydrostatic pressure and

μ is the molecular viscosity of fluid.

$$\boxed{\begin{aligned} \mu_{air} &\approx 1.8\times 10^{-5}\ \text{Pa.s} \\ \mu_{water} &\approx 1.0\times 10^{-3}\ \text{Pa.s} \end{aligned}}$$

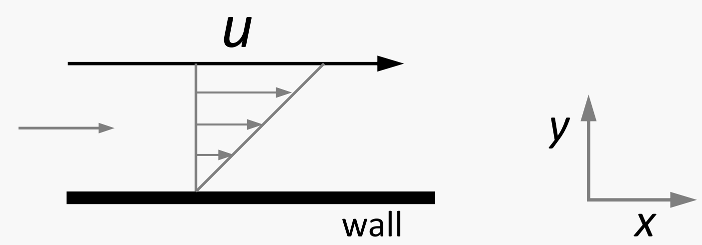
Navier-Stokes Equations

For shear stresses (3D):

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$



For 1-D flow (laminar):

$$\tau = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) = \mu\frac{\partial u}{\partial y}$$

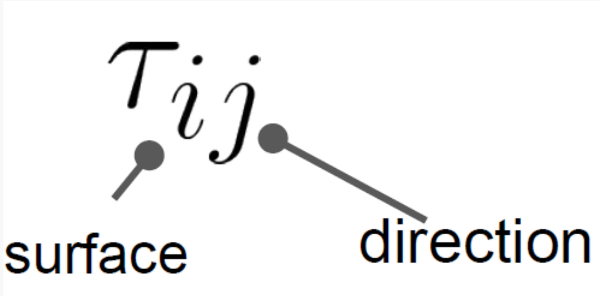
$$\Rightarrow \tau = \tau_{yx} = \mu\frac{\partial u}{\partial y}$$

Newton's law of viscosity

Navier-Stokes Equations

Stress tensor: (9 elements)

$$\tau_{ij}=\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}$$



$$\Rightarrow \tau_{ij}=\begin{bmatrix}-p+2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right)\\ \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & -p+2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right)\\ \mu\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) & -p+2\mu\dfrac{\partial w}{\partial z}\end{bmatrix}$$

Navier-Stokes Equations

Now, use these stress-deformation relations in **NS equation in x-direction:**

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

$$=\rho f_x+\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$=\rho f_x-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial z^2}$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\left(\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial x^2}\right)+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial z^2}$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)+\mu\frac{\partial^2 u}{\partial z^2}$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

Navier-Stokes Equations

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\ \mu\frac{\partial}{\partial x}\cancelto{=0}{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad ;\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

(continuity equation for incompressible flow)

$$\boxed{\Rightarrow\ \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho f_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)}$$

Navier-Stokes Equations

Finally, the set of **Navier-Stokes** equations for **viscous incompressible flow** in Cartesian Coordinates (x,y,z) are:

$$x: \quad \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho f_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$y: \quad \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho f_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$z: \quad \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho f_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Non-linear partial differential equations

2^{nd} order partial differential equations

Since the Navier-Stokes (NS) equations are nonlinear, second-order partial differential equations, these are not manageable for exact mathematical solutions except in a few simplified fluid flow cases (ME 323).

Numerical solution is a must to solve much complicated partial differential equations (NS equations). This opens a broad horizon of mechanical engineering research (**C**omputational **F**luid **D**ynamics-**CFD**).

Navier-Stokes Equations

Millennium Prize Problems

seven unsolved problems in mathematics that were stated by the Clay Mathematics Institute on May 24, 2000.

A correct solution to any of the problems results in a **US$1 million** prize being awarded by the institute to the discoverer(s).

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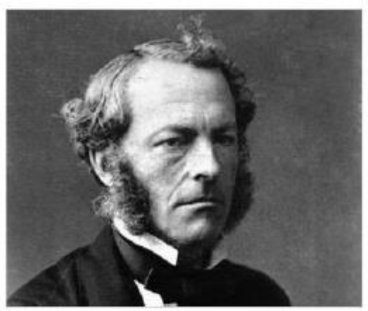
Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). Public Domain

This problem is: Unsolved

Rules:
Rules for the Millennium Prizes

Related Documents:
Official Problem Description

Related Links:
Lecture by Luis Cafarelli

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Millennium Problems

Yang–Mills and Mass Gap
Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis
The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem
If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation
This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture
The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture
In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture
Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

https://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation

Navier-Stokes Equations

Navier-Stokes equation in short form:

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right) \vec{V} = \vec{g} - \frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla^2 \vec{V}$$

body force is shown to be replaced by gravity only.

Navier-Stokes Equations

There are **4 unknowns** (p, u, v, and w), so the solution of these variables are possible when set of NS equations combined with continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$x: \quad \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho f_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$y: \quad \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho f_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$z: \quad \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho f_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Euler equation

In case of **inviscid incompressible flow (μ = 0)**, the Navier-Stokes equations reduce to **Euler equation** as:

$$x: \quad \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho f_x$$

$$y: \quad \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho f_y$$

$$z: \quad \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho f_z$$

In compact form using vector notation; **Euler equation** comes as:

$$\boxed{\rho\frac{D\vec{V}}{Dt}=-\nabla p+\rho\vec{f}}$$

pressure gradient

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\vec{f} = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

Problem

An incompressible velocity field is defined as

$$u = a(x^2 - y^2)$$

$$v = ?$$

$$w = b$$

where a and b are constants. What must be the form of the velocity component v be?

$$Ans: \quad v = -2axy$$